



Static and Dynamic Properties of a One-Dimensional Spin-1/2 System

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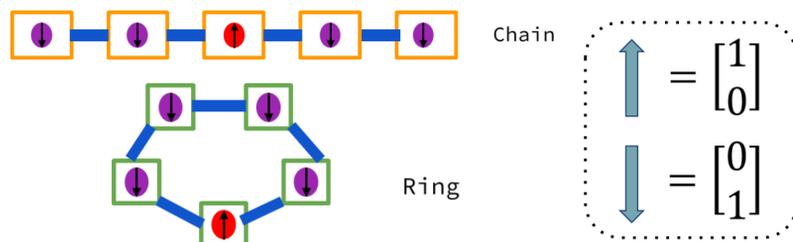


Objectives

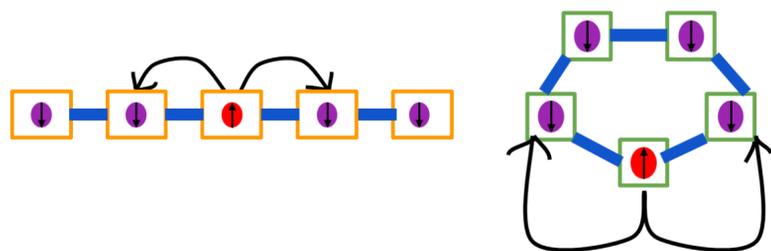
- Description of a 1D spin 1/2 model
- Study of its time evolution (dynamics)
- Illustration of the spread of an excitation through the system

1D spin 1/2 Lattice System & Formation

- On each site there is a spin 1/2 (up or down in z direction)
- Chain or ring formations determine geometry between sites



- Particle pointing upwards is the excitation
- Interactions happen between neighboring sites and the two endpoints (in the ring formation)
- As a result the excitation hops to either side of the neighboring site



Spin 1/2 Hamiltonian Equation

- J is the coupling strength and gives the energy scale
- L is the number of sites
- The first term gives the energy of each site
- The second is called the Flip-Flop term because it moves the excitation along the chain

$$\hat{H} = \sum_{n=1}^L \left[\frac{\epsilon_n}{2} \sigma_n^z \right] + \frac{J}{2} \sum_{n=1}^{L-1} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y \right]$$

- The Flip-Flop term moves the excitation along the chain

$$\frac{J}{2} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) | \uparrow \downarrow \rangle = \frac{J}{2} | \downarrow \uparrow \rangle$$

$$\frac{J}{2} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y) | \downarrow \uparrow \rangle = \frac{J}{2} | \uparrow \downarrow \rangle$$

Pauli Matrices

- The Pauli matrices are the operators that describe the different types of particle interactions

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- The sigma x matrix flips the spin

$$\sigma^x | \uparrow \rangle \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = | \downarrow \rangle$$

$$\sigma^x | \downarrow \rangle \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = | \uparrow \rangle$$

- The sigma y matrix flips the spin and has an additional i term

$$\sigma^y | \uparrow \rangle \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = +i | \downarrow \rangle$$

$$\sigma^y | \downarrow \rangle \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i | \uparrow \rangle$$

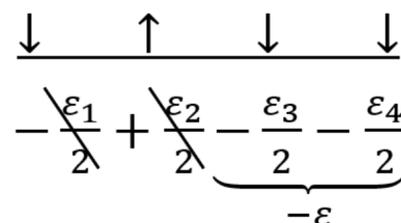
- The sigma z matrix gives the energy of each site (down becomes negative)

$$\frac{\epsilon_1}{2} \sigma_1^z | \uparrow \rangle = \frac{\epsilon_1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = + \frac{\epsilon_1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = + \frac{\epsilon_1}{2} | \uparrow \rangle$$

$$\frac{\epsilon_1}{2} \sigma_1^z | \downarrow \rangle = \frac{\epsilon_1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \frac{\epsilon_1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \frac{\epsilon_1}{2} | \downarrow \rangle$$

Site Basis-Vectors

- $L=4$ with 1 excitation $\uparrow \downarrow \downarrow \downarrow, \downarrow \uparrow \downarrow \downarrow, \downarrow \downarrow \uparrow \downarrow, \downarrow \downarrow \downarrow \uparrow$
- States where on each site the spin either points up or down (in the z direction)
- The Zeeman energies of all sites are equal, $\epsilon_n = \epsilon$



Hamiltonian in a Matrix Form

- Diagonal elements are the sum the energies of each site
- Off-diagonal elements are produced by the Flip-Flop term
- Tridiagonal Matrix (real and symmetric)

Hamiltonian in a Matrix Form (cont.)

Example: open chain with $L=4$ sites and 1 excitation

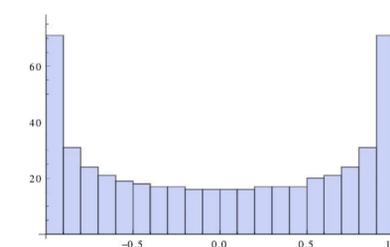
$$\begin{matrix} \uparrow \downarrow \downarrow & \downarrow \uparrow \downarrow & \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow \\ \uparrow \downarrow \downarrow & \downarrow \uparrow \downarrow & \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow \\ \downarrow \uparrow \downarrow & \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow \\ \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow & \downarrow \downarrow \downarrow \uparrow \end{matrix} \begin{bmatrix} -\epsilon & J/2 & 0 & 0 \\ J/2 & -\epsilon & J/2 & 0 \\ 0 & J/2 & -\epsilon & J/2 \\ 0 & 0 & J/2 & -\epsilon \end{bmatrix}$$

- We diagonalize the Hamiltonian matrix to get all eigenvalues, E_n and eigenstates, $|\psi_n\rangle$
- For each energy level (eigenvalue) there is one corresponding eigenstate
- Each eigenstate is a superposition of several site-basis vectors

Density of States

- $L=20$ and 1 excitation

Numerical:



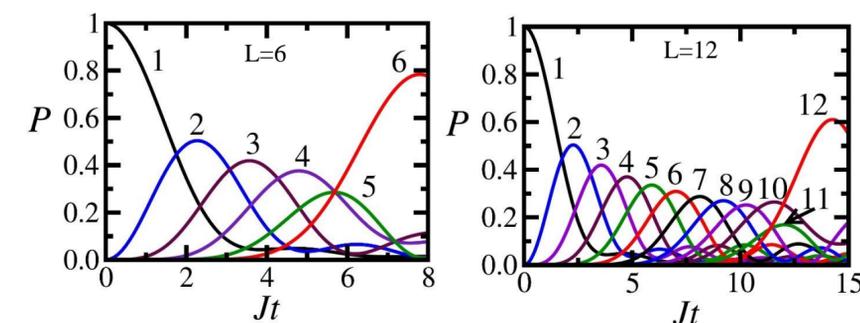
Analytical:

$$\rho(\epsilon) = \frac{1}{\pi \sqrt{J^2 - \epsilon^2}}$$

- Initially, we place the excitation on the first site. Later in time it has a probability of being found on other sites of the chain
- The spreading of an excitation along the chain is a typical property of quantum mechanics

Schrödinger's equation: Evolution of the state:

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \quad |\Psi(t)\rangle = \sum_{n=1}^L C_n e^{-iE_n t} |\psi_n\rangle$$



Conclusion

- Spin 1/2 systems are prototype quantum many-body systems.
- They can be used to introduce undergraduate students to the properties of quantum mechanics and a variety of current subjects of interest, such as: quantum phase transition, metal-insulator transition, quantum chaos, thermalization, and quantum computing.